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Second order tangent bundles of infinite dimensional manifolds

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Abstract

The second order tangent bundle T^2M of a smooth manifold M consists of the equivalent classes of curves on M that agree up to their acceleration. It is known [Analele Stiintifice ale Universitatii Al. I. Cuza 28 (1982) 63] that in the case of a finite n-dimensional manifold M, T^2M becomes a vector bundle over M if and only if M is endowed with a linear connection. Here we extend this result to M modeled on an arbitrarily chosen Banach space and more generally to those Fréchet manifolds which can be obtained as projective limits of Banach manifolds. The result may have application in the study of infinite dimensional dynamical systems.

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1. Introduction

The notion of the second order tangent bundle T^2M of a smooth manifold M as the equivalent classes of curves on M that agree up to their acceleration, seems to be a natural generalization of the classical notion of tangent bundle TM of M. However, the definition of a vector bundle structure on T^2M not only is not as evident as in the case of tangent bundles but, in fact, is not always possible.

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Dodson and Radivoiovici [1] proved that in the case of a finite n-dimensional manifold M, a vector bundle structure on T^2M can be well defined if and only if M is endowed with a linear connection. More precisely, T^2M becomes then and only then a vector bundle over M with structure group the general linear group $GL(2n; \mathbb{R})$ and, therefore, a $3n$-dimensional manifold.

In this paper, we extend the aforementioned results to a wide class of infinite dimensional manifolds. First we consider a manifold M modeled on an arbitrarily chosen Banach space $\mathbb{E}$. Using the Vilms [8] point of view for connections on infinite dimensional vector bundles and a new formalism, we generalize Dodson and Radivoiovici's main theorem by proving that T^2M can be thought of as a Banach vector bundle over M with structure group $GL(\mathbb{E}\times\mathbb{E})$ if and only if M admits a linear connection.

Taking one step further, we study also the case of Fréchet (non-Banach) modeled manifolds. In this framework things proved much more complicated since there are intrinsic difficulties with Fréchet spaces. For example, pathological general linear groups, which do not even admit reasonable topological group structures, put in question even the way of defining vector bundles. However, by restricting ourselves to those Fréchet manifolds which can be obtained as projective limits of Banach manifolds (see e.g. [2]), it is possible to endow T^2M with a vector bundle structure over M with structure group a new topological (and in a generalized sense Lie) group which replaces the pathological general linear group of the fibre type. This construction is equivalent with the existence on M of a specific type of linear connection characterized by a generalized set of Christoffel symbols.

The result should in principle be of interest in the study of infinite dimensional dynamical systems, since important geometrical and physical properties are normally associated with curvature properties of trajectories in the system state space and this curvature is controlled by the second order tangent structure. The new result provides conditions on when the space of accelerations is simplified by the existence of a connection.

2. Preliminaries

In this section we summarize all the necessary preliminary material that we need for a self contained presentation of our paper.

Let M be a C^∞-manifold modeled on a Banach space $\mathbb{E}$ and $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ a corresponding atlas. The latter gives rise to an atlas $\{(\pi_M^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha\in I}$ of the tangent bundle TM of M with

$$\Psi_\alpha : \pi_M^{-1}(U_\alpha) \to \psi_\alpha(U_\alpha) \times \mathbb{E} : [c, x] \mapsto (\psi_\alpha(x), (\psi_\alpha \circ c)'(0)),$$

where $[c, x]$ stands for the equivalence class of a smooth curve c of M with $c(0) = x$ and $(\psi_\alpha\circ c)'(0) = [\mathrm{d}(\psi_\alpha\circ c)(0)](1)$. The corresponding trivializing system of $T(TM)$ is denoted by $\{(\pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)), \tilde{\Psi}_\alpha)\}_{\alpha\in I}$.

Adopting the formalism of Vilms [8], a connection on M is a vector bundle morphism:

$$D : T(TM) \to TM$$

with the additional property that the mappings $\omega_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \to \mathcal{L}(\mathbb{E}, \mathbb{E})$ defined by the local forms of D:

$$D_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \psi_\alpha(U_\alpha) \times \mathbb{E}$$

with $D_\alpha := \Psi_\alpha \circ D \circ (\tilde{\Psi}_\alpha)^{-1}$, $\alpha \in I$, via the relation

$$D_\alpha(y, u, v, w) = (y, w + \omega_\alpha(y, u) \cdot v),$$

are smooth. Furthermore, D is a linear connection on M if and only if $\{\omega_\alpha\}_{\alpha\in I}$ are linear with respect to the second variable.

Such a connection D is fully characterized by the family of Christoffel symbols $\{\Gamma_\alpha\}_{\alpha\in I}$, which are smooth mappings

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \to \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{E})),$$

defined by $\Gamma_\alpha(y)[u] = \omega_\alpha(y, u)$, $(y, u) \in \psi_\alpha(U_\alpha) \times \mathbb{E}$.

The requirement that a connection is well defined on the common areas of charts of M, yields the Christoffel symbols satisfying the following compatibility condition:

$$\begin{aligned}&\Gamma_\alpha(\sigma_{\alpha\beta}(y))(\mathrm{d}\sigma_{\alpha\beta}(y)(u))[\mathrm{d}(\sigma_{\alpha\beta}(y))(v)] + (\mathrm{d}^2\sigma_{\alpha\beta}(y)(v))(u)\\ &\quad = \mathrm{d}\sigma_{\alpha\beta}(y)((\Gamma_\beta(y)(u))(v))\end{aligned} \tag{1}$$

for all $(y, u, v) \in \psi_\alpha(U_\alpha) \times \mathbb{E} \times \mathbb{E}$, and d, d^2 stand for the first and the second differential, respectively. For further details and the relevant proofs we refer to [8].

In the sequel we give some hints for a class of Fréchet manifolds that we will employ in the last section of this note. Let $\{M^i; \varphi^{ji}\}_{i,j\in\mathbb{N}}$ be a projective system of Banach manifolds modeled on the Banach spaces $\{\mathbb{E}^i\}$, respectively. If we assume that:

(i) the models form also a projective limit $\mathbb{F} = \lim_{\leftarrow} \mathbb{E}^i$,
(ii) for each $x = (x^i) \in M$ there exists a projective system of local charts $\{(U^i, \psi^i)\}_{i\in\mathbb{N}}$ such that $x^i \in U^i$ and the corresponding limit $\lim_{\leftarrow} U^i$ is open in M,

then the projective limit $M = \lim_{\leftarrow} M^i$ can be endowed with a Fréchet manifold structure modeled on $\mathbb{F}$ via the charts $\{(\lim_{\leftarrow} U^i, \lim_{\leftarrow} \psi^i)\}$. Moreover, the tangent bundle TM of M is also endowed with a Fréchet manifold structure of the same type modeled on $\mathbb{F} \times \mathbb{F}$. The local structure now is defined by the projective limits of the differentials of $\{\psi^i\}$ and TM turns out to be an isomorph of $\lim_{\leftarrow} TM^i$. Here we adopt the definition of Leslie [5,6] for the differentiability of mappings between Fréchet spaces. However, the differentiability proposed by Kriegl and Michor [4] is also suited to our study.

3. Tangent bundles of order two for infinite dimensional Banach manifolds

Let M be a smooth manifold modeled on the infinite dimensional Banach space $\mathbb{E}$ and $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ a corresponding atlas. For each $x \in M$ we define the following equivalence relation on $C_x = \{f : (-\varepsilon, \varepsilon) \to M | f \text{ smooth and } f(0) = x, \varepsilon > 0\}$:

$$f \approx_x g \Leftrightarrow f'(0) = g'(0) \quad \text{and} \quad f''(0) = g''(0), \tag{2}$$

where by f' and f'' we denote the first and the second, respectively, derivatives of f:

$$f' : (-\varepsilon, \varepsilon) \to TM : t \mapsto [df(t)](1), \qquad f'' : (-\varepsilon, \varepsilon) \to T(TM) : t \mapsto [df'(t)](1).$$

Definition 3.1. We define the *tangent space of order two* of M at the point x to be the quotient $T_x^2 M = C_x / \approx_x$ and the *tangent bundle of order two* of M the union of all tangent spaces of order 2: $T^2 M := \cup_{x \in M}\ T_x^2 M$.

It is worth noting here that $T_x^2 M$ can be always thought of as a topological vector space isomorphic to $\mathbb{E} \times \mathbb{E}$ via the bijection

$$T_x^2 M \overset{\simeq}{\Leftrightarrow} \mathbb{E} \times \mathbb{E} : [f, x]_2 \mapsto ((\psi_\alpha \circ f)'(0), (\psi_\alpha \circ f)''(0)),$$

where $[f, x]_2$ stands for the equivalence class of f with respect to $\approx_x$. However, this structure depends on the choice of the chart (U_α, ψ_α), hence a definition of a vector bundle structure on $T^2 M$ cannot be achieved by the use of the aforementioned bijections. The most convenient way to overcome this obstacle is to assume that the manifold M is endowed with an additional structure: a linear connection.

Theorem 3.2. *If we assume that a linear connection D is defined on the manifold M, then $T^2 M$ becomes a Banach vector bundle with structure group the general linear group $GL(\mathbb{E} \times \mathbb{E})$.*

Proof. Let $\pi_2 : T^2 M \to M$ be the natural projection of $T^2 M$ to M with $\pi_2([f, x]_2) = x$ and $\{\Gamma_\alpha : \psi_\alpha(U_\alpha) \to \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{E}))\}_{a \in I}$ the Christoffel symbols of the connection D with respect to the covering $\{(U_a, \psi_a)\}_{a \in I}$ of M. Then, for each $\alpha \in I$, we define the mapping $\Phi_\alpha : \pi_2^{-1}(U_\alpha) \to U_\alpha \times \mathbb{E} \times \mathbb{E}$ with

$$\begin{aligned}\Phi_\alpha([f, x]_2) = (x, (\psi_\alpha \circ f)'(0), (\psi_\alpha \circ f)''(0) \\ + \Gamma_\alpha(\psi_\alpha(x))((\psi_\alpha \circ f)'(0))[(\psi_\alpha \circ f)'(0)]).\end{aligned}$$

These are obviously well defined and injective mappings. They are also surjective since every element $(x, u, v) \in U_\alpha \times \mathbb{E} \times \mathbb{E}$ can be obtained through Φ_α as the image of the equivalence class of the smooth curve

$$f : \mathbb{R} \to \mathbb{E} : t \mapsto \psi_\alpha(x) + tu + \tfrac{1}{2}(t^2)(v - \Gamma_\alpha(\psi_\alpha(x))(u)[u]),$$

appropriately restricted in order to take values in $\psi_\alpha(U_\alpha)$. On the other hand, the projection of each Φ_α to the first factor coincides with the natural projection $\pi_2 : pr_1 \circ \Phi_\alpha = \pi_2$. Therefore, the trivializations $\{(U_\alpha, \Phi_\alpha)\}_{a \in I}$ define a fibre bundle structure on $T^2 M$ and we need now to focus on the behavior of the mappings Φ_α on areas of M that are covered by common domains of different charts. Indeed, if (U_α, ψ_α), (U_β, ψ_β) are two such charts, let $(\pi_2^{-1}(U_\alpha), \Phi_\alpha)$, $(\pi_2^{-1}(U_\beta), \Phi_\beta)$ be the corresponding trivializations of $T^2 M$. Taking into account the compatibility condition (1) satisfied by the Christoffel symbols $\{\Gamma_\alpha\}$ we see that:

$$(\Phi_\alpha \circ \Phi_\beta^{-1})(x, u, v) = \Phi_\alpha([f, x]_2),$$

where $(\psi_\beta \circ f)'(0) = u$ and $(\psi_\beta \circ f)''(0) + \Gamma_\beta(\psi_\beta(x))(u)[u] = v$. As a result

$$\begin{aligned}(\Phi_\alpha \circ \Phi_\beta^{-1})(x, u, v) = ((\psi_\alpha \circ \psi_\beta^{-1})(\psi_\beta(x)), \mathrm{d}(\psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta \circ f)(0)(1), \\ \mathrm{d}^2(\psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta \circ f)(0)(1, 1)\end{aligned}$$

$$
\begin{aligned}
&\quad + \Gamma_\alpha((\psi_\alpha \circ \psi_\beta^{-1})(\psi_\beta(x)))(\mathrm{d}(\psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta \circ f)(0)(1)) \\
&\quad \times [\mathrm{d}(\psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta \circ f)(0)(1)] \\
&= (\sigma_{\alpha\beta}(\psi_\beta(x)), \mathrm{d}\sigma_{\alpha\beta}(\psi_\beta(x))(u), \\
&\qquad \mathrm{d}\sigma_{\alpha\beta}(\psi_\beta(x))(\mathrm{d}^2(\psi_\beta \circ f)(0)(1,1)) + \mathrm{d}^2\sigma_{\alpha\beta}(\psi_\beta(x))(u)[u] \\
&\quad + \Gamma_\alpha(\sigma_{\alpha\beta}(\psi_\beta(x)))(\mathrm{d}\sigma_{\alpha\beta}(\psi_\beta(x))(u))[\mathrm{d}\sigma_{\alpha\beta}(\psi_\beta(x))(u)] \\
&= (\sigma_{\alpha\beta}(\psi_\beta(x)), \mathrm{d}\sigma_{\alpha\beta}(\psi_\beta(x))(u), \\
&\qquad \mathrm{d}\sigma_{\alpha\beta}(\psi_\beta(x))(\mathrm{d}^2(\psi_\beta \circ f)(0)(1,1)) + \Gamma_\beta(\psi_\beta(x))(u)[u]) \\
&= (\sigma_{\alpha\beta}(\psi_\beta(x)), \mathrm{d}\sigma_{\alpha\beta}(\psi_\beta(x))(u), \mathrm{d}\sigma_{\alpha\beta}(\psi_\beta(x))(v)),
\end{aligned}
$$

where by $\sigma_{\alpha\beta}$ we denote the diffeomorphisms $\psi_\alpha \circ \psi_\beta^{-1}$. Therefore, the restrictions to the fibres

$$\Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1} : \mathbb{E} \times \mathbb{E} \to \mathbb{E} \times \mathbb{E} : (u, v) \mapsto (\Phi_\alpha \circ \Phi_\beta^{-1})|_{\pi_2^{-1}(x)}(u, v)$$

are linear isomorphisms and the mappings:

$$T_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathcal{L}(\mathbb{E} \times \mathbb{E}, \mathbb{E} \times \mathbb{E}) : x \mapsto \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1}$$

are smooth since $T_{\alpha\beta} = (\mathrm{d}\sigma_{\alpha\beta} \circ \psi_\beta) \times (\mathrm{d}\sigma_{\alpha\beta} \circ \psi_\beta)$ holds for each $\alpha, \beta \in I$.

As a result, T^2M is a vector bundle over M with fibres of type $\mathbb{E} \times \mathbb{E}$ and structure group $GL(\mathbb{E} \times \mathbb{E})$. Moreover, T^2M is isomorphic to $TM \times TM$ since both bundles are characterized by the same cocycle $\{(\mathrm{d}\sigma_{\alpha\beta} \circ \psi_\beta) \times (\mathrm{d}\sigma_{\alpha\beta} \circ \psi_\beta)\}_{\alpha,\beta \in I}$ of transition functions. □

Remark 3.3. Note that in the case of finite dimensional manifolds the vector bundle structure obtained in Theorem 3.2 coincides with that defined by Dodson and Radivoiovici, since the corresponding transition functions are identical (see [1, Corollary 2]), although we based ours on a different-totally coordinate free-approach.

We conclude this section by proving that the converse also of Theorem 3.2 holds.

Theorem 3.4. *Let M be a smooth manifold modeled on the Banach space $\mathbb{E}$. If the second order tangent bundle T^2M of M admits a vector bundle structure, with fibres of type $\mathbb{E} \times \mathbb{E}$, isomorphic to the product of vector bundles $TM \times TM$, then a linear connection can be defined on M.*

Proof. Let $\{(\pi_2^{-1}(U_\alpha), \Phi_\alpha)\}_{\alpha \in I}$ be a trivializing cover of T^2M which, according to the hypothesis, restricted to the fibres $\pi_2^{-1}(x) \simeq \pi_M^{-1}(x) \times \pi_M^{-1}(x)$ will have the form: $\Phi_{\alpha,x} = \Phi_{\alpha,x}^1 \times \Phi_{\alpha,x}^2$, where $\Phi_{\alpha,x}^1$ and $\Phi_{\alpha,x}^2$ will be linear isomorphisms from $\pi_M^{-1}(x)$ to $\mathbb{E}$. Then, we may construct a chart (U, ψ_α) of M such that $d_x\psi_\alpha(f'(0)) = \Phi_\alpha^1([f, x]_2)$. Indeed, if (U, ψ) is an arbitrarily chosen chart of M with $U \subseteq U_\alpha$, we may define ψ_α as the composition of ψ with $\Phi_{\alpha,x}^1 \circ (d_x\psi)^{-1}$. Based on these charts we define the Christoffel symbols of the desired connection as follows:

$$\Gamma_\alpha(y)(u, u) = \Phi_{\alpha,x}^2([f, x]_2) - (\psi_\alpha \circ f)''(0), \quad y \in \psi_\alpha(U_\alpha),$$

where f is the curve of M that generates the vector u with respect to the chart ψ_α. The remaining values of $\Gamma_\alpha(y)$ on elements of the form (u, v) with $u \neq v$ are automatically defined if we demand $\Gamma_\alpha(y)$ to be bilinear. These mappings satisfy the necessary compatibility condition (1) since the trivializations $\{(\pi_2^{-1}(U_\alpha), \Phi_\alpha)\}_{\alpha\epsilon I}$ agree, via the transition functions of T^2M, on all common areas of their domains, and, thus, give rise to a linear connection on M. □

4. The Fréchet case

As we have seen in the previous section, the definition of a vector bundle structure on the tangent bundle of order two is always possible for Banach modeled manifolds endowed with a linear connection. However, if we take one step further by considering a manifold M modeled on a Fréchet (non-Banach) space $\mathbb{F}$, then things prove to be much more complicated due to intrinsic difficulties with these types of topological vector spaces.

More precisely, the pathological structure of the general linear groups $GL(\mathbb{F})$, $GL(\mathbb{F}\times\mathbb{F})$ which do not even admit non-trivial topological group structures raises the question of whether *any* possible vector bundle structure can be defined on T^2M. On the other hand, the fact that the space of continuous linear mappings between Fréchet spaces does not remain in the same category of topological vector spaces, as well as the lack of a general solvability theory of differential equations on $\mathbb{F}$, turns the study of connections of the manifold M into a very complicated issue.

In this section, by employing a new methodology—which has already been proven successful for classical tangent and frame bundles (see [3,7])—we develop a vector bundle structure for the second order tangent bundles of a certain class of Fréchet manifolds: those which can be obtained as projective limits of Banach manifolds.

To this end, let M be a smooth manifold modeled on the Fréchet space $\mathbb{F}$. Taking into account that the latter *always* can be realized as a projective limit of Banach spaces $\{\mathbb{E}^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$ (i.e. $\mathbb{F} \cong \lim_{\leftarrow} \mathbb{E}^i$), we assume that the manifold itself is obtained as the limit of a projective system of Banach modeled manifolds $\{M^i; \varphi^{ji}\}_{i,j\in\mathbb{N}}$ in the sense described in the preliminaries. Then, we obtain the following proposition.

Proposition 4.1. *The second order tangent bundles $\{T^2M^i\}_{i\in\mathbb{N}}$ form also a projective system with limit* (*set-theoretically*) *isomorphic to T^2M.*

Proof. For every pair of indices (i, j) with $j \geq i$, we define the mapping:

$$g^{ji} : T^2M^j \to T^2M^i : [f, x]_2^j \mapsto [\phi^{ji} \circ f, \phi^{ji}(x)]_2^i,$$

where the brackets $[\cdot,\cdot]_2^j$, $[\cdot,\cdot]_2^i$ denote the classes of the equivalence relation (2) on M^j, M^i, respectively. We easily check that g^{ji} is always well defined, since two equivalent curves f, g on M^j will give

$$d^{(n)}\phi^{ji}(f^{(n)}(0)) = (\phi^{ji} \circ f)^{(n)}(0) = (\phi^{ji} \circ g)^{(n)}(0) = d^{(n)}\phi^{ji}(g^{(n)}(0)), \quad n = 0, 1, 2,$$

where $d^{(1)}\phi^{ji} : TM^j \to TM^i$ stands for the first and $d^{(2)}\phi^{ji} : T(TM^j) \to T(TM^i)$ for the second differential of ϕ^{ji}.

On the other hand, the relations $g^{ik} \circ g^{ji} = g^{jk}$ $(j \geq i \geq k)$, readily obtained from the corresponding ones for $\{\varphi^{ji}\}_{i,j\in\mathbb{N}}$, ensures that $\{T^2M^i; g^{ji}\}_{i,j\in\mathbb{N}}$ is a projective system. Based now on the canonical projections $\phi^i : M \to M^i$ of M, we define

$$F^i : T^2M \to T^2M^i : [f, x]_2 \to [\phi^i \circ f, \phi^i(x)]_2^i \quad (i \in \mathbb{N}).$$

Since $g^{ji} \circ F^j = F^i$ holds for all $j \geq i$, we obtain the mapping

$$F = \lim_{\leftarrow} F^i : T^2M \to \lim_{\leftarrow} (T^2M^i) : [f, x]_2 \to ([\phi^i \circ f, \phi^i(x)]_2^i)_{i\in\mathbb{N}}.$$

This is an injection because $F([f, x]) = F([g, x])$ gives

$$d^{(n)}\phi^i(f^{(n)}(0)) = (\phi^i \circ f)^{(n)}(0) = (\phi^i \circ g)^{(n)}(0) = d^{(n)}\phi^i(g^{(n)}(0)), \quad n = 0, 1, 2,$$

and, therefore, $f^{(n)}(0) = g^{(n)}(0)$ $(n = 0, 1, 2)$ since $TM \equiv \lim_{\leftarrow} TM^i$ and $T(TM) \equiv \lim_{\leftarrow} T(TM^i)$ with corresponding canonical projections $\{d\phi^i\}_{i\in\mathbb{N}}$ and $\{d^{(2)}\phi^i\}_{i\in\mathbb{N}}$, respectively.

On the other hand, F is also surjective since for every element $a = ([f^i, x^i]_2^i)_{i\in\mathbb{N}} \in \lim_{\leftarrow} (T^2M^i)$ we see that:

$$[\phi^{ji} \circ f^j, \phi^{ji}(x^j)]_2^i = [f^i, x^i]_2^i \quad \text{for } j \geq i, \tag{3}$$

thus $x = (x^i) \in M = \lim_{\leftarrow} M^i$. Moreover, if $(U = \lim_{\leftarrow} U^i, \psi = \lim_{\leftarrow} \psi^i)$ is a projective limit chart of M through x and $(\pi_M^{-1}(U) = \lim_{\leftarrow} \pi_{M^i}^{-1}(U^i), \Psi = T\psi = \lim_{\leftarrow} T\psi^i)$, $(\pi_{TM}^{-1}(\pi_M^{-1}(U)) = \lim_{\leftarrow} \pi_{TM^i}^{-1}(\pi_{M^i}^{-1}(U^i)), \tilde{\Psi} = T(T\psi) = \lim_{\leftarrow} T(T\psi^i))$ the corresponding charts of TM, $T(TM)$, respectively, we obtain:

$$\begin{aligned}
&((\psi^i \circ \phi^{ji} \circ f^j)(0), T\psi^i((\phi^{ji} \circ f^j)'(0))) \\
&\quad = ((\psi^i \circ f^i)(0), T\psi^i((f^i)'(0))) \Rightarrow (\rho^{ji}((\psi^j \circ f^j)(0)), T\psi^i(T\phi^{ji}((f^j)'(0)))) \\
&\quad = ((\psi^i \circ f^i)(0), T\psi^i((f^i)'(0))).
\end{aligned}$$

As a result, the elements $u = ((\psi^i \circ f^i)(0))_{i\in\mathbb{N}}$, $v = ((\psi^i \circ f^i)'(0))_{i\in\mathbb{N}}$ belong to $\mathbb{F} \cong \lim_{\leftarrow} \mathbb{E}^i$. Similarly, relations (3) ensure that $(\phi^{ji} \circ f^j)''(0) = (f^i)''(0)$ which via the charts of $T(TM)$ defined above give $T(T\psi^i)((\phi^{ji} \circ f^j)''(0)) = T(T\psi^i)((f^i)''(0))$ or, equivalently, $\rho^{ji}((\psi^j \circ f^j)''(0)) = (\psi^i \circ f^i)''(0)$, for $j \geq i$. Therefore, $w = ((\psi^i \circ f^i)''(0))_{i\in\mathbb{N}}$ belongs also to $\mathbb{F} \cong \lim_{\leftarrow} \mathbb{E}^i$. Considering now the curve h of $\mathbb{F}$ with $h(t) = u + t \cdot v + (t^2/2) \cdot w$, as well as the corresponding one f of M with respect to the chart $(U = \lim_{\leftarrow} U^i, \psi = \lim_{\leftarrow} \psi^i)$, we may check that

$$(\phi^i \circ f)(0) = \phi^i(x) = x^i = f^i(0),$$

$$\begin{aligned}
(\phi^i \circ f)'(0) &= (\psi^{i-1} \circ \rho^i \circ h)'(0) = T\psi^{i-1}((\rho^i \circ h)'(0)) = T\psi^{i-1}(\rho^i(v)) \\
&= T\psi^{i-1}((\psi^i \circ f^i)'(0)) = (f^i)'(0),
\end{aligned}$$

$$(\phi^i \circ f)''(0) = (\psi^{i-1} \circ \rho^i \circ h)''(0) = T(T\psi^{i-1})((\rho^i \circ h)''(0)) = T(T\psi^{i-1})(\rho^i(w))$$
$$= T(T\psi^{i-1})((\psi^i \circ f^i)''(0)) = (f^i)''(0)$$

for all indices i, j with $j \geq i$. As a result, the curves $\phi^i \circ f$, f^i are equivalent on M^i and $F([f, x]_2) = ([f^i, x^i]_2^i)_{i\in\mathbb{N}} = a$.

By this means, we ensure that the mapping F is the desired isomorphism which turns T^2M, $\lim_{\leftarrow}(T^2M^i)$ to isomorphic sets. □

Based on the last result, next we define a vector bundle structure on T^2M by means of a certain type of linear connection on M. The problems concerning the structure group of this bundle (discussed earlier) are overcome by the replacement of the pathological $GL(\mathbb{F} \times \mathbb{F})$ by the new topological (and in a generalized sense smooth Lie) group:

$$\mathcal{H}^0(\mathbb{F} \times \mathbb{F}) := \left\{ (l^i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} GL(\mathbb{E}^i \times \mathbb{E}^i) : \lim_{\leftarrow} l^i \text{ exists} \right\}.$$

To be more specific, $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})$ is a topological group being isomorphic to the projective limit of the Banach–Lie groups

$$\mathcal{H}_i^0(\mathbb{F} \times \mathbb{F}) := \left\{ (l^1, l^2, \ldots, l^i)_{i\in\mathbb{N}} \in \prod_{k=1}^{i} GL(\mathbb{E}^k \times \mathbb{E}^k) : \rho^{jk} \circ l^j = l^k \circ \rho^{jk} \ \ (k \leq j \leq i) \right\}.$$

On the other hand, it can be considered as a generalized Lie group via its embedding in the topological vector space $\mathcal{L}(\mathbb{F} \times \mathbb{F})$. Using these notations we obtain the following theorem.

Theorem 4.2. *If a Fréchet manifold* $M = \lim_{\leftarrow} M^i$ *is endowed with a linear connection* D *that can be also realized as a projective limit of connections* $D = \lim_{\leftarrow} D^i$*, then* T^2M *is a Fréchet vector bundle over* M *with structure group* $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})$.

Proof. Following the terminology established in Section 2, we consider $\{(U_\alpha = \lim_{\leftarrow} U_\alpha^i, \psi_\alpha = \lim_{\leftarrow} \psi_\alpha^i)\}_{\alpha\in I}$ an atlas of M. Each linear connection D^i ($i \in \mathbb{N}$), which is naturally associated to a family of Christoffel symbols $\{\Gamma_\alpha^i : \psi_\alpha^i(U_\alpha^i) \to \mathcal{L}(\mathbb{E}^i, \mathcal{L}(\mathbb{E}^i, \mathbb{E}^i))\}_{\alpha\in I}$, ensures that T^2M^i is a vector bundle over M^i with fibres of type $\mathbb{E}^i$. This structure, as already presented in Theorem 3.2, is defined by the trivializations:

$$\Phi_\alpha^i : (\pi_2^i)^{-1}(U_\alpha^i) \to U_\alpha^i \times \mathbb{E}^i \times \mathbb{E}^i$$

with

$$\Phi_\alpha^i([f, x]_2^i) = (x, (\psi_\alpha^i \circ f)'(0), (\psi_\alpha^i \circ f)''(0) + \Gamma_\alpha^i(\psi_\alpha^i(x))((\psi_\alpha^i \circ f)'(0))[(\psi_\alpha^i \circ f)'(0)]); \quad \alpha \in I.$$

Taking into account that the families of mappings $\{g^{ji}\}_{i,j\in\mathbb{N}}$, $\{\varphi^{ji}\}_{i,j\in\mathbb{N}}$, $\{\rho^{ji}\}_{i,j\in\mathbb{N}}$ are connecting morphisms of the projective systems $T^2M = \lim_{\leftarrow}(T^2M^i)$, $M = \lim_{\leftarrow} M^i$, $\mathbb{F} = \lim_{\leftarrow} \mathbb{E}^i$, respectively, we check that the projections $\{\pi_2^i : T^2M^i \to M^i\}_{i\in\mathbb{N}}$ satisfy

$$\varphi^{ji} \circ \pi_2^j = \pi_2^i \circ g^{ji} \quad (j \geq i),$$

and the trivializations $\{\Phi_\alpha^i\}_{i\in\mathbb{N}}$

$$(\varphi^{ji} \times \rho^{ji} \times \rho^{ji}) \circ \Phi_\alpha^j = \Phi_\alpha^i \circ g^{ji} \quad (j \geq i).$$

As a result

$$\pi_2 = \varprojlim \pi_2^i : T^2M \to M,$$

exists and is a surjective mapping

$$\Phi_\alpha = \varprojlim \Phi_\alpha^i : \pi_2^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F} \times \mathbb{F} \quad (\alpha \in I)$$

is smooth, as a projective limit of smooth mappings, and its projection to the first factor coincides with π_2.

On the other hand, the restrictions of Φ_α to any fibre $\pi_2^{-1}(x)$ is a bijection since $\Phi_{\alpha,x} := pr_2 \circ \Phi_\alpha|_{\pi_2^{-1}(x)} = \lim_{\leftarrow} (pr_2 \circ \Phi_\alpha^i|_{(\pi_2^i)^{-1}(x)})$.

The crucial part of our construction, however, concerns the corresponding transition functions $\{T_{\alpha\beta} = \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1}\}_{\alpha,\beta\in I}$. These can be considered as taking values in the generalized Lie group $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})$, since $T_{\alpha\beta} = \epsilon \circ T_{\alpha\beta}^*$, where $\{T_{\alpha\beta}^*\}_{\alpha,\beta\in I}$ are the smooth mappings

$$T_{\alpha\beta}^* : U_\alpha \cap U_\beta \to \mathcal{H}^0(\mathbb{F} \times \mathbb{F}) : x \mapsto (pr_2 \circ \Phi_\alpha^i|_{(\pi_2^i)^{-1}(x)})_{i\in\mathbb{N}},$$

and ϵ is the natural inclusion

$$\epsilon : \mathcal{H}^0(\mathbb{F} \times \mathbb{F}) \to \mathcal{L}(\mathbb{F} \times \mathbb{F}) : (l^i)_{i\in\mathbb{N}} \mapsto \varprojlim l^i.$$

Summarizing, we have proved that T^2M admits a vector bundle structure over M with fibres of type $\mathbb{F} \times \mathbb{F}$ and structure group $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})$. Moreover, this bundle is isomorphic to $TM \times TM$ since we may check that they have identical transition functions:

$$T_{\alpha\beta}(x) = \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1} = (\mathrm{d}(\psi_a \circ \psi_\beta^{-1}) \circ \psi_\beta)(x) \times (\mathrm{d}(\psi_a \circ \psi_\beta^{-1}) \circ \psi_\beta)(x). \qquad \square$$

We conclude this paper by proving that also the converse of Theorem 4.2 is true.

Theorem 4.3. *If T^2M is an $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})$-Fréchet vector bundle over M isomorphic to $TM \times TM$, then M admits a linear connection which can be realized as a projective limit of connections.*

Proof. By the hypothesis, the vector bundle structure on T^2M would be defined by a family of trivializations $\{\Phi_\alpha : \pi_2^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F} \times \mathbb{F}\}_{\alpha\in I}$ which will be realized as projective limits of corresponding trivializations $\Phi_\alpha^i : (\pi_2^i)^{-1}(U_\alpha^i) \to U_\alpha^i \times \mathbb{E}^i \times \mathbb{E}^i$ of T^2M^i ($i \in \mathbb{N}$) so that the transition functions $\{T_{\alpha\beta}\}_{\alpha,\beta\in I}$ of T^2M take their values in $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})$. As a result, every factor-bundle T^2M^i will be a vector bundle isomorphic to $TM^i \times TM^i$ and, according to Theorem 3.4, a linear connection D^i can be defined on M^i with Christoffel symbols satisfying

$$\Gamma^i_\alpha(y)(u^i, u^i) = (pr_3 \circ \Phi^i_\alpha)([f^i, x]^i_2) - (\psi^i_\alpha \circ f^i)''(0),$$

where f^i is the curve of M^i generating the vector u^i with respect to the chart ψ^i_α.

We may check then that $\lim_{\leftarrow} (\Gamma^i_\alpha(y^i)(u^i))$ exists for all (y^i), $(u^i) \in \mathbb{F} = \lim_{\leftarrow} \mathbb{E}^i$, thus the connections $\{D^i\}_{i\in\mathbb{N}}$ form a projective system with corresponding limit the desired linear connection $D = \lim_{\leftarrow} D^i$ on M. $\square$

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